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Nonstandard Variational Calculus with Applications to Classical Mechanics. 1. An Existence Criterion

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Using the framework of nonstandard analysis, I find the discretized version of the Euler–Lagrange equation for classical dynamical systems and discuss the existence of an extremum for a given functional in variational calculus. Some results related to the Cauchy existence theorem are obtained and discussed with various examples.

1. INTRODUCTION

In the recent decades the heuristic idea of infinite and infinitesimal numbers has obtained formal rigor due to the work of Robinson.[(1)] He essentially proved that the field of real numbers **R** can be considered as a proper subset of a new field **R*** which contains both numbers larger than any positive real number (infinite) and positive numbers smaller than any positive real number (infinitesimal).

Many papers have been produced on this subject, especially from the mathematical point of view. In the last 20 years, some physical applications also have appeared in the literature (see refs. 2–8 and references therein).

On the other hand, the enormous improvement of computer-aided techniques for solving physical and/or mathematical problems has prompted many workers to develop different discretization procedures in many different contexts (see ref. 9 and references therein for some applications). A natural requirement is that any "discretized" model must coincide with its continuous counterpart whenever the discretization parameter h goes to zero, so that

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the differences between the two models are negligible for h small enough. Nonstandard analysis (*NSA*) provides a rather natural framework where these kinds of problems can be discussed.

In this paper we investigate the possibility of using *NSA* to compute the extremum of a given functional $J[y] \equiv \int_a^b F(x, y, y')\,dx$ satisfying the boundary conditions $y(a) = A$ and $y(b) = B$. We will show that this is possible. The action principle will produce a set of *ns-finite* (see below) algebraic equations whose solution differs from the standard one, that is, the one obtained by solving the usual Euler–Lagrange differential equation related to the functional $J[q]$,[10] for an infinitesimal quantity.

Furthermore we will prove that these equations, which we will call nonstandard Euler–Lagrange equations (NSELE), coincide with the equations one would obtain by discretizing directly the standard Euler–Lagrange equation (SELE).

Finally, many examples are discussed to show the applicability of the method and its limits.

We end this Introduction with a bit of notation: here and in the following we indicate by *s* or *standard* any quantity which "lives" in $\mathbf{R}$, while *ns* or *nonstandard* will be used for numbers, functions or whatever has a meaning only in $\mathbf{R}^*$. For instance, we will say that the number a is s-finite if there exists a positive real r such that $|a| < r < \infty$. We refer to refs. 1 and 11 for more information on *NSA*.

2. VARIATIONAL CALCULUS AND THE DISCRETIZATION OF THE ACTION INTEGRAL

Let $F(x, y, z)$ be a function with all the first and second partial derivatives continuous. Our main task will be to find, among all the functions $y(x)$ at least continuously differentiable such that $y(a) = A$ and $y(b) = B$, those for which the functional

$$J[y] \equiv \int_a^b F\,(x, y, y')\,dx \tag{2.1}$$

has an extremum.

This is a well-known problem, widely discussed in the literature (e.g., refs. 10, 12, 13), which is solved once the SELE

$$F_y - \frac{d}{dx} F_{y'} = 0 \tag{2.2}$$

is solved, and the solutions coincide. In particular, in ref. 10 it is proved, for instance, that a solution of (2.2) satisfying the correct boundary conditions is also an extremum of the functional $J[y]$.

In classical mechanics, in order to describe a point particle, one takes $F(x, y, z)$ to be the Lagrangian of the system, $L(t, q, \dot{q})$. Here t is the time, $q = q(t)$ is the position of the point particle, and $\dot{q}(t)$ is its time derivative. We refer to refs. 14 and 15 for many details on the Lagrangian approach to classical mechanics. In this way one obtains a different form of the second principle of the dynamics

$$\mathbf{F} = m\mathbf{a}$$

where $\mathbf{F}$ is the force applied to the particle and $\mathbf{a}$ its acceleration. Finding a solution of equation (2.2) with $y(a) = A$ and $y(b) = B$ is now a typical example of a boundary value problem.

Many textbooks of calculus of variations discuss the possibility of finding an approximate solution of the extremum problem. Various possibilities are proposed: the one we consider here is known as the *finite difference method*.[10,16] Taking this method as our starting point, we will show in this paper that NSA makes it possible to translate the SELE into an infinite set of algebraic equations whose solution differs in most cases from the standard solution of the differential equation (2.2) by an infinitesimal quantity. Of course this means that the nonstandard version of the finite difference method is really very close, if not equivalent, to the original standard approach and it is not merely an approximation.

In *NSA* an s-bounded function f: $[a, b] \to \mathbf{R}$ is (Riemann) integrable if there exists s-finite

$$st\Big[\sum_{a \leq jh \leq b} f(jh)h\Big]$$

and this quantity does not depend on the choice of the infinitesimal h.[11] We recall that, given a hyperreal $a \in \mathbf{R}^*$, its standard part, $st[a]$, is the (unique) real number such that $a - st[a] \in \mathbf{R}^0$, where $\mathbf{R}^0$ indicates the subset of $\mathbf{R}^*$ of all the infinitesimal numbers.

Let us assume therefore that the function $L(t, q, z)$ is integrable in $[t_i, t_f]$ with respect to t, where q and z are functions of t. Let furthermore Δ belong to $\mathbf{N}^*$, the set of hypernatural numbers, and $\eta \equiv (t_f - t_i)/\Delta$. We introduce the following partition of the interval $[t_i, t_f]$: $t_0 = t_i$, $t_\Delta = t_f$, and $t_k = t_{k-1} + \eta$ for all k, between 1 and $\Delta - 1$. Therefore, if Δ is s-infinite, by definition

$$J[q] \equiv \int_{t_i}^{t_f} L(t, q(t), \dot{q}(t))dt = st\left[\eta \sum_{k=0}^{\Delta-1} L(t_k, q_k, \dot{q}_k)\right] \tag{2.3}$$

where $q_k \equiv q(t_k)$ and $\dot{q}_k \equiv \dot{q}(t_k)$. Of course, due to the hypothesis of integrability of the function L, the above result is independent of the choice of η.

The "time" derivative $\dot{q}_k(t)$ can be defined in different ways: we, could for instance, use central differences (see ref. 17, p. 192), but we would rather use the following more natural definition:

$$\dot{q}_k(t) \equiv st\left[\frac{q_{k+1}(t) - q_k(t)}{\eta}\right] = st\left[\frac{q(t_{k+1}) - q(t_k)}{\eta}\right] \qquad (2.4)$$
$$= st\left[\frac{q(t_k + \eta) - q(t_k)}{\eta}\right]$$

The reason central differences are used in numerical integration of differential equations is that they allow a numerical error of $O(\eta^2)$ instead of $O(\eta)$.[17] We will come back to this point in the last section. Here we only want to note that, since $\eta \in \mathbf{R}^0$, this difference will not be crucial because in both cases we will conclude that this error belongs to $\mathbf{R}^0$.

Even if the equality in equation (2.3) holds only if $\eta \in \mathbf{R}^0$, we will always consider first η as a small quantity not in $\mathbf{R}^0$ and we will take η in $\mathbf{R}^0$ only at a second time. Therefore, it is interesting to estimate the difference between the action $J[q]$ and its approximation $\eta\ \Sigma_{k=0}^{\Delta-1}\ L(t_k, q_k, \dot{q}_k)$. In particular, we want to show that, under appropriate regularity conditions on the function L, this difference will be proportional to η. Furthermore, we will also show that the substitution $\dot{q}_k(t) \to [q(t_k + \eta) - q(t_k)]/\eta$ does not distroy this estimate. Namely, we will prove that, calling

$$q_{k\eta}(t) \equiv \frac{q(t_k + \eta) - q(t_k)}{\eta} \qquad (2.5)$$

the estimate

$$\left|\int_{t_i}^{t_f} L(t, q(t), \dot{q}(t))dt - \eta \sum_{k=0}^{\Delta-1} L(t_k, q_k, q_{k\eta})\right| = O(\eta)$$

still holds true. This estimate seems to suggest that the nonstandard Euler finite-difference method can be conveniently applied; in fact, our double approximation [integration $\to$ sum, $\dot{q}_k(t) \to q_{k\eta}(t)$] does not modify $J[q]$ for more than a constant times η. However, this is not a proof of the validity of this approach, which can only be ensured by some *a priori* estimate on the difference between the solution of the SELE and the solution of the system we are going to find below, (2.14). Nevertheless, it is obviously interesting to know what happens to $J[q]$ after the discretization procedure, since it may be interpreted as a measure of the "closeness" of the standard and nonstandard techniques.

Once this estimate is proved we will proceed as for the canonical approximate computation of the extremum of a functional,[10,16] and we will get a system of $\Delta - 1$ equations with $\Delta - 1$ variables. We will discuss, using various examples, the utility of this system, focusing in particular on the role of η. We will analyze with care what happens if η is taken in $\boldsymbol{R}^0$.

Using a slightly different notation than ref. 11, we will say that a function f: $\boldsymbol{R}^* \to \mathbf{R}^*$ is ns-continuous if for all $x \approx y$, $f(x) \approx f(y)$. By $x \approx y$ we mean that x and y have the same standard part or, equivalently, that they belong to the same monad. It is worth recalling that in the same reference it is shown that if f: $\boldsymbol{R} \to \boldsymbol{R}$ and $x \in \boldsymbol{R}$ are both standard, then f is ns-continuous in x if and only if f is continuous in x. Many examples are also discussed in order to show that this equivalence does not hold if f or x (or both) is nonstandard.

Let us define the following quantities:

$$J[q_1, \dot{q}_1, \ldots, q_{\Delta-1}, \dot{q}_{\Delta-1}] \equiv \sum_{k=0}^{\Delta-1} L(t_k, q_k, \dot{q}_k) \cdot \eta \tag{2.6}$$

$$J[q_1, \ldots, q_{\Delta-1}] \equiv \sum_{k=0}^{\Delta-1} L(t_k, q_k, q_{k\eta}) \cdot \eta \tag{2.7}$$

The following proposition holds:

Proposition 1. Let $L(t, q(t), \dot{q}(t))$ be a differentiable function for which a positive constant M exits such that $\left|(d/dt)\, L(t, q(t), \dot{q}\,(t))\right| < M$, $\forall t \in [t_i, t_f]$. Then

$$\left|J[q] - J\,[q_1, \dot{q}_1, \ldots, q_{\Delta-1}, \dot{q}_{\Delta-1}]\right| \leq \frac{\eta}{2} M(t_f - t_i) \tag{2.8}$$

Moreover, if L is ns-continuous in $\dot{q}(t)$ and η is taken in $\boldsymbol{R}^0$, then

$$st[J[q_1, \dot{q}_1, \ldots, q_{\Delta-1}, \dot{q}_{\Delta-1}]] = st[J[q_1, \ldots, q_{\Delta-1}]] \tag{2.9}$$

Proof. The first statement can be deduced with an easy adaptation of some well-known techniques of Riemann integration. We essentially split the integral from t_i to t_f into a sum of integrals from t_k *to* $t_k + \eta$, for $k = 0, 1, \ldots, \Delta - 1$. Each of these contributions is then approximated using Taylor's formula around $\eta = 0$. For instance,

$$\int_{t_0}^{t_1} L(t, q(t), \dot{q}(t))dt$$
$$= L(t_0, q(t_0), \dot{q}(t_0))$$
$$+ \frac{d}{d\mu} L(\mu + t_0, q(\mu + t_0), \dot{q}(\mu + t_0))\Big|_{\mu=\xi_1} \frac{\eta^2}{2}$$

where $t_0 \leq \xi_1 \leq t_1$.

We get

$$J[q] = \sum_{k=0}^{\Delta-1} L(t_k, q_k, \dot{q}_k) \cdot \eta + \sum_{k=1}^{\Delta} \frac{d}{d\mu} L(t_0 + \mu, q(t_0 + \mu), \dot{q}\,(t_0 + \mu))\Big|_{\mu=\xi_k} \frac{\eta^2}{2}$$

where, for all $k = 0, 1, 2, 3, \ldots, \Delta - 1$, we have $t_k \leq \xi_{k+1} \leq t_{k+1}$. This equation, definition (2.6), and the hypothesis on L give estimate (2.8).

To prove the second part of the proposition we first observe that the ns-continuity of L in $\dot{q}(t)$ implies, since $st[\dot{q}(t)] = st[\{q(t+\eta) - q(t)\}/\eta]$, that

$$st[L(t, q(t), \dot{q}(t))] = st\left[L\left(t, q(t), \frac{q(t+\eta) - q(t)}{\eta}\right)\right] \tag{2.10}$$

Using the linearity of the standard part, from definitions (2.6) and (2.7), we get

$$st[J[q_1, \dot{q}_1, \ldots, \dot{q}_{\Delta-1}] - J[q_1, \ldots, q_{\Delta-1}]] = st\left[\sum_{k=0}^{\Delta-1} L(t_k, q_k, \dot{q}_k\,(t)) \cdot \eta - \sum_{k=0}^{\Delta-1} L(t_k, q_k, q_{k\eta}) \cdot \eta\right] = 0$$

This follows from the equality (2.10), which holds for all t in $[t_i, t_f]$, and from the fact that $\eta \cdot \Sigma_{k=0}^{\Delta-1} = \eta \cdot \Delta = t_f - t_i$, which is an s-finite quantity.

Remarks. 1. We stress that equation (2.9) essentially says that, like the function $J[q_1, \dot{q}_1, \ldots, q_{\Delta-1}, \dot{q}_{\Delta-1}]$, also $J[q_1, \ldots, q_{\Delta-1}]$ differs from $J[q]$ for something which is proportional to η (eventually raised to some positive power). Therefore it is reasonable to consider $J[q_1, \ldots, q_{\Delta-1}]$ as a good approximation of $J[q]$, or, in other words, to put $J[q] = st[J[q_1, \ldots, q_{\Delta-1}]]$.

2. We are not requiring to L to be a standard map. However, this is what usually happens, for instance, in most physical situations. The hypothesis of the previous proposition show that it can also be applied to the case in which L is an ns-function.

3. If L is the Lagrangian of a physical conservative system,[14,15] then using the SELE, we get

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\,\dot{q}\right) + \frac{\partial L}{\partial t} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\,\dot{q}\right)$$

The last equality follows from the fact that, since the energy must be conserved,[14] the Lagrangian cannot depend explicitly on the time t. In further detail, if L describes a particle with unit mass in a conservative potential we get $(\partial L/\partial \dot{q})\dot{q} = 2T$, where T is the kinetic energy of the particle. Therefore the hypothesis of the proposition simply says that the kinetic energy cannot have a divergent time derivative. This is physically more than reasonable since T is usually a regular function of t (more than $C^{(1)}$, so that its derivative is certainly bounded in any bounded time interval.

Let us now look for the critical points of $J[q_1, \ldots, q_{\Delta-1}]$. These are the points that satisfy the set of equations

$$\frac{\partial J[q_1, \ldots, q_{\Delta-1}]}{\partial q_k} = 0, \qquad k = 1, 2, \ldots, \Delta - 1 \tag{2.11}$$

which in NSA are rewritten as

$$st\left[\frac{J[q_1, \ldots, q_k + \tau, \ldots, q_{\Delta-1}] - J[q_1, \ldots, q_{\Delta-1}]}{\tau}\right] = 0,$$
$$k = 1, 2, \ldots, \Delta - 1 \tag{2.12}$$

where $\tau \in \boldsymbol{R}^0$. We observe that all the contributions in the sum defining $J[q_1, \ldots, q_{\Delta-1}]$ but two cancel each other in the above difference. This is obviously due to the presence of q_k in only two terms of the sum in (2.7). After some easy calculations we see that equations (2.12) can be explicitly written as

$$st\left[\frac{\eta}{\tau}\left\{L\left(t_{k-1}, q_{k-1}, \frac{q_k + \tau - q_{k-1}}{\eta}\right) + L\left(t_k, q_k + \tau, \frac{q_{k+1} - (q_k + \tau)}{\eta}\right)\right.\right.$$
$$\left.\left. - L\left(t_{k-1}, q_{k-1}, \frac{q_k - q_{k-1}}{\eta}\right) - L\left(t_k, q_k, \frac{q_{k+1} - q_k}{\eta}\right)\right\}\right] = 0,$$
$$\forall\, k = 1, 2, \ldots, \Delta - 1 \tag{2.13}$$

The presence of η in the numerator would trivialize all the equations if it is taken in $\boldsymbol{R}^0$. Therefore, in order not to obtain the equality "0 = 0" for all k, we simply forget η. This will be justified in the following and is related to the fact that we have in mind to consider at a first stage η as a finite (noninfinitesimal) quantity, and then consider it as an element of $\boldsymbol{R}^0$ only at the very end. This explains why we are allowed at this stage to divide equation (2.13) by η.

We conclude finally that the NSELE are the following set of $\Delta - 1$ equations

$$st\left[\frac{1}{\tau}\left\{L\left(t_{k-1}, q_{k-1}, \frac{q_k + \tau - q_{k-1}}{\eta}\right) + L\left(t_k, q_k + \tau, \frac{q_{k+1} - (q_k + \tau)}{\eta}\right)\right.\right.$$
$$\left.\left. - L\left(t_{k-1}, q_{k-1}, \frac{q_k - q_{k-1}}{\eta}\right) - L\left(t_k, q_k, \frac{q_{k+1} - q_k}{\eta}\right)\right\}\right] = 0,$$

$$\forall\, k = 1, 2, \ldots, \Delta - 1 \tag{2.14}$$

The solution of this system of equations, $(q_1, q_2, \ldots, q_{\Delta-1})$, gives information on the solution $q(t)$ of the SELE: in fact the polygon passing through the points (t_0, q_0), (t_1, q_1), (t_2, q_2),. . . . , (t_Δ, q_Δ) approximates the solution $q(t)$ of the SELE, with the required boundary conditions, as soon as η is taken small but real. If $\eta \in \boldsymbol{R}^0$, then this polygonal *is* the solution of the classical problem, in the sense that for any $t_k \in [t_i, t_f]$ the value of q_k is expected to have the same standard part of $q(t_k)$. We will came back to this point in the last section.

The following proposition shows that the system (2.14) is exactly the discretization of the SELE with its boundary conditions.

Proposition 2. Let us consider the functional $J[q] \equiv \int_{t_i}^{t_f} L(t, q(t), \dot{q}(t))\, dt$, and its discretization $J[q_1, \ldots, q_{\Delta-1}]$, with $q_0 = q_i$ and $q_\Delta = q_f$ for $\Delta \in \boldsymbol{N}$, s-finite. Then the system (2.14), if $st[\eta] = 0$ (that is, in the limit $\Delta \to \infty$), produces the Euler–Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0, \qquad q(t_i) = q_i, \quad q(t_f) = q_f \tag{2.15}$$

Proof. Let us first take η as a small noninfinitesimal quantity. Therefore, considering the first and the third terms in (2.14), we have, with after simple algebra,

$$st\left[\frac{1}{\tau}\left(L\left(t_{k-1}, q_{k-1}, \frac{q_k + \tau - q_{k-1}}{\eta}\right) - L\left(t_{k-1}, q_{k-1}, \frac{q_k - q_{k-1}}{\eta}\right)\right)\right]$$

$$= st\left[\frac{L\left(t_{k-1}, q_{k-1}, \dfrac{q_k + \tau - q_{k-1}}{\eta}\right) - L\left(t_{k-1}, q_{k-1}, \dfrac{q_k - q_k - 1}{\eta}\right)}{(q_k + \tau - q_{k-1})/\eta - (q_k - q_{k-1})/\eta}\right.$$

$$\left.\times \frac{\dfrac{q_k + \tau - q_{k-1}}{\eta} - \dfrac{q_k - q_{k-1}}{\eta}}{\tau}\right]$$

$$= st\left[\frac{L(t_{k-1}, q_{k-1}, q_{k-1\eta} + \tau/\eta) - L(t_{k-1}, q_{k-1}, q_{k-1\eta})}{q_{k-1\eta} + \tau/\eta - q_{k-1\eta}}\right.$$

$$\left.\times \frac{q_{k-1\eta} + \tau/\eta - q_{k-1\eta}}{\tau}\right]$$

$$= \frac{1}{\eta} st\left[\frac{L(t_{k-1}, q_{k-1}, q_{k-1\eta} + \tau/\eta) - L(t_{k-1}, q_{k-1}, q_{k-1\eta})}{q_{k-1\eta} + \tau/\eta - q_{k-1\eta}} \right]$$

$$= \frac{1}{\eta} \frac{\partial}{\partial q_{k-1\eta}} L(t_{k-1}, q_{k-1}, q_{k-1\eta})$$

where the derivative must be obviously understood in the nonstandard sense.

Analogously, we can handle the second and the fourth terms of (2.14), to obtain

$$st\left[\frac{1}{\tau} \left(L\left(t_k, q_k + \tau, \frac{q_{k+1} - (q_k + \tau)}{\eta} \right) - L\left(t_k, q_k, \frac{q_{k+1} - q_\kappa}{\eta} \right) \right) \right]$$

$$= \frac{\partial}{\partial q_k} L(t_k, q_k, q_{k\eta}) - \frac{1}{\eta} \frac{\partial}{\partial q_{k\eta}} L(t_k, q_k, q_{k\eta})$$

Therefore system (2.14) is equivalent to

$$\frac{\partial}{\partial q_k} L(t_k, q_k, q_{k\eta}) - \frac{1}{\eta} \left(\frac{\partial}{\partial q_{k\eta}} L(t_k, q_k, q_{k\eta}) - \frac{\partial}{\partial q_{k-1\eta}} L(t_{k-1}, q_{k-1}, q_{k-1\eta}) \right) = 0$$

$$\forall k = 1, 2, \ldots, \Delta - 1$$

Following refs. 10 and 16, we conclude that, when $\eta \in \mathbf{R}^0$ (or, in standard language, when $\Delta \to \infty$), these equations "converge" to equation (2.15).

Remark. The above statement was not obvious *a priori*. It shows that the discretization procedure can be applied with the same conclusions both to the action $J[q]$ and directly to the SELE. Therefore the extremum of $J[q_1, \ldots, q_{\Delta-1}]$ is also a solution of the discretized Euler–Lagrange equations, and conversely.

3. EXAMPLES

Some of the examples of this section have a certain relevance in physics, others will only be considered from a mathematical point of view.

We start considering a certain class of "Lagrangians" $L(t, q(t), \dot{q}(t))$: let $f(x)$ be any function (standard or not standard) of the (standard or not standard) variable x. Then we put

$$L_f(t, q(t), \dot{q}(t)) \equiv \dot{q}^2(t) + f(q(t)) \tag{3.1}$$

which does not depend explicitly on t. Some regularity conditions must be sought for f if we wish to require, for instance, continuity in q of L. It is easy to see that for any such L_f Proposition 1 applies as far as $\dot{q}(t)$ is s-bounded.[11] This Lagrangian is of particular interest in physics since it contains a kinetic term (but a factor 1/2) so that it really describes a classical particle in a potential $f(q)$. Its limitation on $\dot{q}(t)$, which is satisfied, for instance, whenever the time interval is a compact set and $\dot{q}(t)$ is continuous, simply says that the kinetic energy of the particle must be finite at any time.

Analogously one can show also that Lagrangians of the form

$$L_h(t, q(t), \dot{q}(t)) \equiv \alpha\dot{q}^n(t) + h(q(t), t) \tag{3.2}$$

for any real α and any natural n satisfy the hypothesis of the same proposition. However, for $n \neq 2$, L_h has no evident physical meaning.

Let us now start with the examples.

Example 1. $L(q(t), \dot{q}(t)) = \dot{q}^2(t) + 2q(t)$.

We want to find the extremum $q(t)$ for the following functional $J[q]$:

$$J[q] = \int_0^1 (\dot{q}^2(t) + 2q(t))\, dt$$

with the boundary conditions $q(0) = q(1) = 0$. Using the SELE, we see that $q(t)$ must be a solution of $\ddot{q}(t) = 1$, with $q(0) = q(1) = 0$. The solution obviously exists and is unique: $q(t) = (t^2 - t)/2$.

We will now find the form of system (2.14) and we will show that these NSELE allow us to conclude that a unique solution exists.

First we observe that L fits into the class (3.1), with $f(q) = 2q$, so that the discretization of $J[q]$ is controlled.

Due to the given boundary conditions we have $q_0 = q(0) = 0$ and $q_\Delta = q(1) = 0$. Furthermore, $\forall k = 1, 2, \ldots, \Delta - 1$,

$$L\left(q_{k-1}, \frac{q_k + \tau - q_{k-1}}{\eta}\right) - L\left(q_{k-1}, \frac{q_k - q_{k-1}}{\eta}\right) = \frac{\tau^2}{\eta^2} + 2\tau\,\frac{q_k - q_{k-1}}{\eta^2}$$

and

$$L\left(q_k + \tau, \frac{q_{k+1} - (q_k + \tau)}{\eta}\right) - L\left(q_k, \frac{q_{k+1} - q_k}{\eta}\right) = \frac{\tau^2}{\eta^2} - 2\tau\,\frac{q_{k+1} - q_k}{\eta^2} + 2\tau$$

so that the final set of equations (2.14) is

$$2q_k - q_{k-1} - q_{k+1} = -\eta^2, \qquad 1 \leq k \leq \Delta - 1 \tag{3.3}$$

with $q_0 = q_\Delta = 0$. (We remind the reader that, for the moment, η is small, but not in $\mathbf{R}_0$, so that the number of equations $\Delta - 1$ is standard-finite.)

It is interesting to verify the statement of Proposition 2 using this example: we can easily see that these equations are exactly the ones we would get by discretizing the SELE for this model, $\ddot{q}(t) = 1$, with the substitution

$$q(t) \to q_k, \qquad \dot{q}(t) \to \frac{q_{k+1} - q_k}{\eta}, \qquad \ddot{q}(t) \to \frac{q_{k+1} - 2q_k + q_{k-1}}{\eta^2} \tag{3.4}$$

which is a canonical procedure in numerical analysis. We can rewrite equations (3.3) in a matricial form:

$$M_{\Delta-1}\, Q = -\eta^2 U \tag{3.5}$$

where $M_{\Delta-1}$ is a $(\Delta - 1) \times (\Delta - 1)$ matrix, and Q and U are column vectors with $\Delta - 1$ components:

$$M_{\Delta-1} \equiv \begin{pmatrix} 2 & -1 & 0 & . & . & 0 & 0 \\ -1 & 2 & -1 & 0 & . & . & 0 \\ 0 & -1 & 2 & -1 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & -1 & 2 & -1 \\ 0 & 0 & 0 & . & 0 & -1 & 2 \end{pmatrix},$$

$$Q \equiv \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ . \\ . \\ q_{\Delta-2} \\ q_{\Delta-1} \end{pmatrix}, \qquad U \equiv \begin{pmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ 1 \\ 1 \end{pmatrix}$$

$M_{\Delta-1}$ is a tridiagonal matrix, which is the typical form of matrix obtained in discretization procedures of any second-order linear differential equation.

This topic is widely discussed in almost any book of numerical analysis,[18,19] where also criteria for the invertibility of such matrices are given.

Following formula (A.2) in the Appendix, we are able to prove that $\forall\Delta$ we have det $M_{\Delta-1} = \Delta$. The proof is very simple and is obtained by induction on Δ. Let $\Delta = 2$. Then det $M_{\Delta-1} =$ det $M_1 = 2$, so the first step is proved.

Let us now suppose that det $M_{n-1} = n$, $\forall n \leq \Delta$, and let us try to compute det M_Δ. Using (A.2), we deduce that

$$\det M_\Delta = 2 \det M_{\Delta-1} - 1^2 \det M_{\Delta-2} = 2\Delta - (\Delta - 1) = \Delta + 1$$

so that our claim is proved. The same result also can be obtained using directly formula (A.7) for det $M_{\Delta-1}$ We deduce therefore that M_Δ can be inverted for all natural Δ. Of course, due to the increasing behavior of det M_Δ, the invertibility of the matrix is still true even when η is taken in $\mathbf{R}^0$, so that the solution of our system exists and is unique:

$$Q = -\eta^2 M_{\Delta-1}^{-1} U$$

We will discuss in the next section the validity of this solution, that is, in which sense Q is related to the solution of the SELE.

Example 2. $L(t, q(t), \dot{q}(t)) = \dot{q}^2(t) + 2tq(t) + q^2(t)$.
Our functional is now

$$J[q] = \int_0^1 (\dot{q}^2(t) + 2tq(t) + q^2(t))\, dt$$

and we look for the extremum of $J[q]$ satisfying the boundary conditions $q(0) = q(1) = 0$. Again, the SELE gives the unique solution $q(t) = e(e^t - e^{-t})/(e^2 - 1) - t$.

We observe that the Lagrangian we are considering belongs to the class (3.2), with $n = 2$, $\alpha = 1$, and $h(q(t), t) = 2tq(t) + q^2(t)$, so that even in this example the dicretization procedure of $J[q]$ is under control.

System (2.14) becomes here

$$q_k(2 + \eta^2) - q_{k-1} - q_{k+1} = -\eta^2 t_k \tag{3.6}$$

with $k = 1, 2, 3, \ldots, \Delta - 1$, and with the boundary conditions $q_0 = q_\Delta = 0$. This is just the same system we obtain by discretizing the SELE, as asserted in Proposition 2.

Again, using a matricial notation, we can write

$$L_{\Delta-1}\, Q = -\eta^2 T \tag{3.7}$$

where

$$L_{\Delta-1} \equiv \begin{pmatrix} 2+\eta^2 & -1 & 0 & . & . & 0 & 0 \\ -1 & 2+\eta^2 & -1 & 0 & . & . & 0 \\ 0 & -1 & 2+\eta^2 & -1 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & -1 & 2+\eta^2 & -1 \\ 0 & 0 & . & . & 0 & -1 & 2+\eta^2 \end{pmatrix}$$

$$Q \equiv \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ . \\ . \\ q_{\Delta-2} \\ q_{\Delta-1} \end{pmatrix}, \qquad T \equiv \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ . \\ . \\ t_{\Delta-2} \\ t_{\Delta-1} \end{pmatrix}$$

It is worthwhile to observe that the matrix L_Δ changes when Δ changes, not only in the number of its rows and columns, but also since the value of η is proportional to the inverse of Δ.

Again the existence and uniqueness of the solution is a matter of invertibility of a matrix, the matrix L_Δ. The existence of L_Δ^{-1} is ensured by Proposition A1, which gives a sufficient condition for a given matrix to have a nonzero determinant. Essentially it states that a symmetric tridiagonal matrix can be inverted if the absolute value of any diagonal element is greater than or equal to the sum of the absolute values of the other elements in the same row. Furthermore, this inequality is required to be *strict* for at least one row.

Of course, whatever the meaning of η, we always have $2+\eta^2 \geq |-1| + |-1|$. Furthermore, the elements of the first and of the last rows satisfy the strict inequality. In order to discuss the existence of L_Δ^{-1} we could also compute explicitly the determinant of L_Δ using equation (A.5), but this alternative procedure is certainly more involved than the one discussed above, which easily follows from Proposition A1.

Therefore also in this example the solution exists and is unique. Again, as one can easily see, the conclusion is totally independent of the nature of η, and whether or not it belongs to $\mathbf{R}^0$.

Example 3. $L(t, q(t), \dot{q}(t)) = \frac{1}{4}\dot{q}^2(t)t^2 + q(t) + \frac{1}{2}q^2(t)t)$.
Our functional is now

$$J[q] = \int_1^2 \left(\frac{1}{4}\,\dot{q}^2(t)t^2 + q(t) + \frac{1}{2}\,q^2(t)t \right) dt$$

and we look for the extremum of $J[q]$ with the boundary conditions $q(1) = q(2) = 0$. We have chosen this new interval for t since it will allow a simpler discussion of the discretization error; see the last section. The SELE produces the following differential equation:

$$\frac{1}{2}\,\ddot{q}(t)t^2 + \dot{q}(t)t = 1 + q(t)t \tag{3.8}$$

which is a nonhomogeneous linear differential equation with nonconstant coefficients. Due to the form of this equation we cannot easily apply the usual theorem on the existence and unicity of solutions of a differential equation. Therefore we are not sure *a priori* that a solution really exists. We will show that our method easily gives an answer to this question, and actually a positive one.

After some algebraic computation we get the following system of equations:

$$\frac{2q_k - q_{k-1} - q_{k+1}}{2\eta^2}\,t_k^2 + \frac{q_k - q_{k-1}}{2} - \frac{q_k - q_{k-1}}{\eta}\,t_k + 1 + q_k t_k = 0 \tag{3.9}$$

for all $k = 1, 2, \ldots, \Delta - 1$, with the boundary conditions $q_0 = q(1) = 0$ and $q_\Delta = q(2) = 0$.

First of all, using relations (3.4), we notice that these equations are, as expected, the discretized version of equation (3.8). We write (3.9) in a matricial form:

$$N_{\Delta-1}Q = -2\eta^2 U \tag{3.10}$$

where the vectors Q and U are the same as in the first example, while

$$N_{\Delta-1} \equiv \begin{pmatrix} A_1 & C_1 & 0 & . & . & 0 & 0 \\ B_2 & A_2 & C_2 & 0 & . & . & 0 \\ 0 & B_3 & A_3 & C_3 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & B_{\Delta-1} & A_{\Delta-2} & C_{\Delta-2} \\ 0 & 0 & . & . & 0 & B_{\Delta-1} & A_{\Delta-1} \end{pmatrix}$$

Here we have defined the following quantities

$$\begin{cases} A_k = 2t_k^2 + 2t_k\eta^2 - 2t_k\eta + \eta^2 \\ B_k = -(t_k - \eta)^2 \\ C_k = -t_k^2 \end{cases}$$

It is worthwhile to notice that $N_{\Delta-1}$ is actually a symmetric tridiagonal matrix since, recalling that $t_k = t_{k-1} + \eta$, it easily follows that $C_k = B_{k+1}$, $\forall k = 1, 2, \ldots, \Delta - 2$.

In order to show that the matrix above can be inverted, we again make use of Proposition A1. We need therefore to verify that $|A_k| \geq |C_k| + |C_{k-1}|$ for all $k = 1, 2, \ldots, \Delta - 1$, where we put $C_0 = C_{\Delta-1} = 0$, and that the strict inequality holds at least for one value of k.

Since $|C_k| = t_k^2$ and $|A_k| = |(t_k - \eta)^2 + t_k^2 + 2t_k\eta| = A_k$ we need to show that $A_k - |C_k| - |C_{k-1}| = A_k + C_k + C_{k-1} \geq 0$. This is certainly verified since, for all $k = 2, 3, \ldots, \Delta - 2$,

$$A_k + C_k + C_{k-1} = 2t_k\eta^2$$

Let us now verify that the strict inequality holds, say, for $k = \Delta - 1$. In this case, in fact, we have

$$A_{\Delta-1} + C_{\Delta-1} + C_{\Delta-2} = A_{\Delta-1} + C_{\Delta-2} = t_{\Delta-1}^2 + 2t_{\Delta-1}\eta^2 > 0$$

strictly, even if $\eta \in \mathbf{R}^0$, since $st[t_{\Delta-1}] = 2$.

Obviously a strict inequality also holds for $k = 1$, for all choices of $\Delta \in \mathbf{N}^*$.

We conclude that it is possible to invert the matrix $N_{\Delta-1}$ whatever Δ is chosen to be, s-finite or ns-finite, so that the solution of (3.10) exists and is unique.

Example 4. $L(q(t), \dot{q}(t)) = \frac{1}{2}(\dot{q}^2(t) - q^2(t))$.

This example shows that our procedure gives nontrivial information even in the situation in which the standard solution exists but is not unique.

Let us consider the functional

$$J[q] = \frac{1}{2}\int_0^{\pi} (\dot{q}^2(t) - q^2(t))\, dt$$

and let us try to find the extremum of $J[q]$ with the boundary conditions $q(0) = q(\pi) = 0$. It is easily seen that $q(t) = B\sin(t)$ is a solution of the SELE for any choice of the constant B. Therefore infinite solutions exist. Our system is already in a matricial form,

$$A_{\Delta-1}\, Q = 0 \tag{3.12}$$

where $A_{\Delta-1}$ is the following $(\Delta - 1) \times (\Delta - 1)$ matrix:

$$A_{\Delta-1} \equiv \begin{pmatrix} 2-\eta^2 & -1 & 0 & . & . & 0 & 0 \\ -1 & 2-\eta^2 & -1 & 0 & . & . & 0 \\ 0 & -1 & 2-\eta^2 & -1 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & -1 & 2-\eta^2 & -1 \\ 0 & 0 & . & . & 0 & -1 & 2-\eta^2 \end{pmatrix}$$

and Q is as in Example 1. In the above matrix η depends on Δ and should be substituted with π/Δ. We also observe that equation (3.12) can be obtained using Proposition 2.

It is easy to see that for Δ finite, det $A_{\Delta-1}$ is in general different from zero, so that the matrix $A_{\Delta-1}$ can be inverted and a unique solution exists, and is obviously $Q = 0$. This existence follows from explicit computation and not from the use of Proposition A1, since its hypotheses are not satisfied by this matrix. We are now interested in understanding what happens if Δ is taken in **N***, not s-finite.

Let us first observe that $|2-\eta^2| < 2|-1|$, so that the determinant of $A_{\Delta-1}$ is given, for all values of Δ, by equation (A.4), and can be put in the form

$$\det A_{\Delta-1} = (-1)^{(\Delta-1)} \frac{\sin[\Delta \arccos(\pi^2/2\Delta^2 - 1)]}{\sin[\arccos(\pi^2/2\Delta^2 - 1)]} \tag{3.13}$$

An easy computation shows that, when $\eta \in \mathbf{R}^0$,

$$st\left[\frac{\det A_{\Delta-1}}{-\pi^2/24\Delta}\right] = 1$$

which implies that det $A_{\Delta-1}$ behaves like $-\pi^2/24\Delta$ when Δ increases in **N***. We conclude therefore that in this "limit." $st[\det A_{\Delta-1}] = 0$, so that the inverse of the matrix fails to exist when $\eta \in \mathbf{R}^0$.

It is worth stressing the relevance of boundary conditions for the existence of a unique solution in this example. It is obvious that the SELE $\ddot{q}(t) + q(t) = 0$, with $q(0) = q(t_f) = 0$, gives the unique solution $q(t) = 0$ whenever $t_f \neq l\pi$, $l \in \mathbb{Z}\setminus\{0\}$. If, on the contrary, $t_f = l\pi$, $l \in \mathbb{Z}\setminus\{0\}$, then the equation admits infinite solutions. Is this duality preserved by our approach?

The answer is that, indeed, we recover this situation also with our framework. The main difference from the previous case is that now η is equal to t_f/Δ so that

$$\det A_{\Delta-1} = (-1)^{(\Delta-1)} \frac{\sin [\Delta \arccos (t_f^2/2\Delta^2 - 1)]}{\sin [\arccos (t_f^2/2\Delta^2 - 1)]}$$

Therefore, after some calculations, we deduce that if $t_f = l\pi$, then $st[\det A_{\Delta-1}] = 0$. If $t_f \neq l\pi$, the situation changes a great deal. We can prove that $\det A_{\Delta-1}$ "diverges" to plus or minus infinity according to the sign of $\sin t_f$, so that the matrix $A_{\Delta-1}$ can be inverted and the solution exists, is unique, and is obviously $Q = 0$.

4. ESTIMATES OF ERRORS AND CONCLUSIONS

In Section 2 we discussed the error we make when the functional $J[q]$ is replaced with its discretized expression $J[q_1, q_2, \ldots, q_{\Delta-1}]$. We showed that this is negligible for small η, and is really infinitesimal for $\eta \in \mathbf{R}^0$. This section is devoted to estimating the differences between the standard solution of the SELE (2.15), $q(t)$, and the solutions $(q_1, q_2, \ldots, q_{\Delta-1})$ of the system of NSELE, (2.14). Even if it is more than reasonable to expect that these solutions converge to $q(t)$ when η is taken in $\mathbf{R}^0$, since in this limit the system (2.14) is nothing but the SELE, an *a priori* estimate is better obtained. This is again a problem widely discussed in numerical analysis, so that we give here some examples and refer to refs. 17–20 for details and other information on this subject.

Since the discretization procedure gives the same set of equations both when applied to $J[q]$ and directly to the SELE, we will consider here as starting points the differential equations instead of the functionals generating them.

4.1. First Model

Let us consider the following class of first-order differential equations:

$$\dot{q}(t) + a(t)q(t) = r(t), \qquad q(a) = q_i, \qquad q(b) = q_f \tag{4.1}$$

where we will suppose that there exists an s-finite positive constant α such that $a(t) \leq -\alpha$ for all $t \in [a, b]$. Furthermore, we will suppose that $a(t)$ is also bounded from below by another s-finite negative constant. Let us define two operators L and L_η as

$$L[y(t)] \equiv \dot{y}(t) + a(t)y(t)$$

for all differentiable functions $y(t)$, and

$$L_\eta[y_i] \equiv \frac{y_{i+1} - y_i}{\eta} + a(t_i)y_i$$

for all sequences $\{y_i\}$. With these definitions equation (4.1) becomes

$$L[q(t)] = r(t), \qquad q(a) = q_i, \qquad q(b) = q_f \tag{4.2}$$

while its discretization is

$$L_\eta[q_j] = r(t_j), \qquad j = 1, 2, \ldots, \Delta - 1, \quad q_0 = q_i, \quad q_\Delta = q_f \tag{4.3}$$

Our problem consists in finding an estimate for the difference $|q(t_j) - q_j|$ for all j, which really expresses the validity of the discretization. To obtain this estimate we will follow the same steps as in ref. 18. We start by defining τ_j by the following equation:

$$L_\eta[q(t_j)] = r(t_j) + \tau_j \tag{4.4}$$

so that, if the solution of equation (4.2) belongs to $C^2([a, b])$, we get

$$\tau_j = L_\eta[q(t_j)] - L[q(t_j)]$$
$$= \frac{q(t_{j+1}) - q(t_j)}{\eta} + a(t_j)q(t_j) - \dot{q}(t_j) - a(t_j)q(t_j) = \frac{\eta}{2}\ddot{q}(\xi_j)$$

where $\xi_j \in [t_j, t_j + \eta]$. Since $\ddot{q}$ is continuous, by assumption, there exists a positive s-finite constant M_2 such that $M_2 = \sup_{t\in[a,b]} |\ddot{q}(t)| < \infty$. Therefore we get

$$\tau \equiv \sup_i |\tau_i| = \frac{\eta}{2}\sup_i |\ddot{q}(\xi_i)| \leq \frac{\eta}{2} M_2$$

Let us now consider the following difference: $e_j \equiv q_j - q(t_j)$. It is easy to see, using (4.3) and (4.4), that e_j must satisfy the following set of equations:

$$e_j(1 - \eta a(t_j)) = e_{j+1} + \eta\tau_j$$

for $j = 1, 2, \ldots, \Delta - 1$, which give, introducing $e \equiv \sup_i |e_i|$ and using our constraint on $a(t)$, the following inequality:

$$e \leq \frac{\tau}{\alpha} \leq \frac{M_2}{2\alpha}\eta$$

Therefore we conclude that if η is taken in $\mathbf{R}^0$, the standard and the nonstandard solutions belong at any time to the same monad.

It is interesting to notice that with a different discretization procedure, the central difference one, in which $\dot{q}(t)$ is replaced with the ratio $(q_{i+1} - q_{i-1})/2\eta$, we can prove that e can be estimated with an $O(\eta^2)$. This loses importance in our nonstandard approach since both η and η^2 have zero standard part.

4.2. Second Model

The second example is described by the following class of second-order differential equations:

$$\ddot{q}(t) - p(t)\dot{q}(t) - g(t)q(t) = r(t), \qquad q(a) = q_i, \qquad q(b) = q_f \quad (4.5)$$

The relevant hypothesis on the functions p and g are the following: both must be s-bounded, so that, for instance, there exists s-finite $P^* = \sup_{t\in[a,b]}|p(t)|$. Furthermore we will assume that there exists a strictly positive quantity g_* such that $g(t) \geq g_*$ for all $t \in [a, b]$.

The main steps are the same as before: we define two operators T and T_η as

$$T[y(t)] \equiv \ddot{y}(t) - p(t)\dot{q}(t) - g(t)q(t)$$

and

$$T_\eta[y_i] \equiv \frac{y_{i-1} - 2y_i + y_{i+1}}{\eta^2} - p(t_i)\frac{y_{i+1} - y_i}{\eta} - g(t_i)y_i$$

so that equation (4.5) becomes

$$T[q(t)] = r(t), \qquad q(a) = q_i, \qquad q(b) = q_f \quad (4.6)$$

and its discretized version is simply

$$T_\eta[q_j] = r(t_j), \qquad j = 1, 2, \ldots, \Delta - 1 \quad (4.7)$$

with $q_0 = q_i$ and $q_\Delta = q_f$. With analogous definitions for τ_i, e_i, τ, e as in the previous model, assuming now that the classical solution $q(t)$ belongs to $C^{(4)}([a, b])$, we get following estimate:

$$e \leq \frac{\eta}{2q_*}\left(P^* M_2 + \frac{M_4\eta}{6}\right)$$

where $M_2 = \sup_{t\in[a,b]}|\ddot{q}(t)|$ and $M_4 = \sup_{t\in[a,b]}|q^{(\mathrm{IV})}(t)|$.

Again, this estimate shows that our procedure can be properly applied to this class of models, in the sense that the standard and the nonstandard solutions differ for infinitesimal quantities at any time.

4.3. Third Model

Another class of models easily controlled is described by the following differential equation:

$$\ddot{q}(t) = f(t, q(t)), \qquad q(a) = \alpha, \qquad q(b) = \beta \quad (4.8)$$

whose discretized version is

$$\frac{q_{i-1} - 2q_i + q_{i+1}}{\eta^2} = f(t_i, q_i), \qquad i = 1, 2, \ldots, \Delta - 1 \quad (4.9)$$

with $y_0 = \alpha$ and $y_\Delta = \beta$. In ref. 19 it is proven that if $q(t)$ is a solution of equation (4.8) and $\{q_i\}$ solves the system (4.9), if there exists s-finite $M = \sup_{t\in[a,b]} \left|q^{(IV)}(t)\right|$, and if $\partial f/\partial y \geq 0$ for all $t \in [a, b]$ and for all y, then

$$\left|q(t_j) - q_j\right| \leq \frac{M\eta^2}{24}(t_j - a)(b - t_j) < \frac{M\eta^2}{24}(b - a)^2 \tag{4.10}$$

for all $j = 1, 2, \ldots, \Delta - 1$, which again proves the equivalence between the standard and the nonstandard solutions.

We now apply the above estimates to the examples discussed in the previous section.

Example 1 is a second-order differential equation which cannot be discussed within the structure of the Model 2 of this section, since the main condition on the function $g(t)$ is not satisfied. However, this example fits into the hypothesis of Model 3 since we have $f(t, q) = 1$ and therefore its derivative with respect to q is identically zero. From this consideration we conclude that whenever $\eta \in \mathbf{R}^0$ then $st[q(t_j) - q_j] = 0$ for any $j = 0, 1, 2, \ldots, \Delta$, so that the solution of the NSA approach coincides with the standard one.

Example 2, again a second-order linear differential equation, fits into both Models 2 and 3. Also for this example we can conclude, therefore, that whenever $\eta \in \mathbf{R}^0$, then $st[q(t_j) - q_j] = 0$ for any $j = 0, 1, 2, \ldots, \Delta$.

Finally, the differential equation of Example 3 can be written, for $t \in [1, 2]$, as

$$\ddot{q}(t) + \frac{2}{t}\dot{q}(t) - \frac{2}{t} = \frac{2}{t^2}$$

It is now easily seen that all the hypotheses of Model 2 are satisfied. In particular, for instance, $g(t) \geq 1$ for all $t \in [1, 2]$. Therefore also in this example standard and nonstandard solutions at any time may differ by not more than an infinitesimal quantity.

We have proposed an alternative method for finding the extremum of a functional $J[q]$ using nonstandard techniques. At first sight the language we speak may appear quite similar to that of numerical analysis and, in fact, many results coming from this branch of mathematics can be put into this new framework. We must remember, in any case, that NSA allows us to obtain rigorous results instead of the approximations one is always forced to deal with in numerical computations.

We also stress that in the examples discussed in the previous section the existence of the unique solution was only a matter of computing a determinant: a deep logical difference with respect to the standard situation!

In this paper we have only discussed the existence of an extremum of the functional $J[q]$; we have not shown how to solve explicitly the system (2.14). We will discuss a general strategy together with many examples in a future paper.[(21)] In particular, among other things, the explicit solutions of the NSELE for Examples 1–4 will be found and the above estimates and conclusions will be explicitly obtained.

We end by noticing that a real limit of this paper is that even if the system (2.14) can give nonlinear equations, all the examples discussed here are related to linear differential equations. This is only a technical limit, since such equations are translated, in our ns-language, into systems of algebraic linear equations which are easily handled. Nonlinear differential equations would produce nonlinear algebraic equations, much more difficult to discuss.

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APPENDIX. MATRIX PROPERTIES

This Appendix is devoted to some results used in this paper related to matrices and determinants. The relevance of these algebraic objects appear clearly throughout the paper, as it follows from our discretization procedure.

We first consider the following matricial equation:

$$A_n \mathbf{x} = \mathbf{d}$$

where A_n is an $n \times n$ tridiagonal matrix and $\mathbf{x}$ and $\mathbf{d}$ are two vectors:

$$A_n = \begin{pmatrix} b_1 & c_1 & 0 & . & . & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & . & . & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & . & 0 & a_n & b_n \end{pmatrix},$$

$$\mathbf{x} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{pmatrix}, \qquad \mathbf{d} \equiv \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \cdot \\ \cdot \\ d_{n-1} \\ d_n \end{pmatrix} \tag{A.1}$$

In ref. 19 it is stated that if A_n is positive definite, that is, if $y^T A_n y > 0$ for any n-dimensional vector y, then the above system can be solved, since det $A_n \neq 0$. In ref. 19 the explicit steps for obtaining the solution are also indicated. We first define $\beta_1 = b_1$ and $\delta_1 = d_1$ and then, for $i = 1, 2, \ldots,$ $n - 1$ we put $\beta_{i+1} = b_{i+1} - (a_{i+1}/\beta_i)c_i$ and $\delta_{i+1} = d_{i+1} - (a_{i+1}/\beta_i)\delta_i$. The solution of the system is now obtained with the substitutions $x_n = \delta_n/\beta_n$ and, for $i = n - 1, n - 2, \ldots 3, 2, 1 \Rightarrow x_i = (\delta_i - c_i x_{i+1})/\beta_i$.

It is now very easy to prove the following useful recursion formula for the determinant of the matrix A_n:

$$\det A_n = b_n \det A_{n-1} - a_n c_{n-1} \det A_{n-2} \tag{A.2}$$

simply computing the determinant det A_n with respect to the last row.

Again ref. 19 shows an interesting result for symmetrical tridiagonal matrices:

Proposition A1. Let A_n be a symmetrical tridiagonal matrix as in (A.1) with $c_i = a_{i+1}$ for all $i = 1, 2, \ldots, n - 1$. Let us define $c_0 = c_n = 0$. Then, if $|b_i| \geq |c_i| + |c_{i-1}|$, $i = 1, 2, \ldots, n$, with the strict inequality holding true at least for one value of i, A_n is positive definite, and therefore det $A_n > 0$.

Another useful result is related to $n \times n$ tridiagonal symmetrical matrices like

$$B_n = \begin{pmatrix} b & a & 0 & . & . & 0 & 0 \\ a & b & a & 0 & . & . & 0 \\ 0 & a & b & a & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & a & b & a \\ 0 & 0 & 0 & . & 0 & a & b \end{pmatrix} \tag{A.3}$$

We define $D_n \equiv \det B_n$. In ref. 22 it is proven that, if $|b| < 2|a|$,

$$D_n = a^n \frac{\sin[(n+1)\theta]}{\sin[\theta]} \tag{A.4}$$

where $\theta = \arccos(b/2a)$. It is an easy exercise to generalize this result to situations in which $|b| > 2|a|$ or $|b| = 2|a|$. In the first case we have

$$D_n = a^n \frac{\sinh[(n+1)\Theta]}{\sinh[\Theta]} \tag{A.5}$$

where $\Theta = (\cosh)^{-1}(b/2a)$.

If $|b| = 2|a|$ it is better to consider two different situations: the first, described by $ba > 0$, corresponds to b and a with the same sign, the second, when $ba < 0$, corresponds to b and a with opposite sign.

If $ba > 0$, we can obtain the value of D_n by taking the standard part of (A.4) when $st[\theta] = 0$. This gives

$$D_n = a^n(n+1) \tag{A.6}$$

If $ba < 0$, we obtain D_n by taking the standard part of (A.4) when $st[\theta] = \pi$. This gives, in turn,

$$D_n = (-a)^n(n+1) \tag{A.7}$$

It is interesting to notice that if $b = 2$ and $a = -1$, then $D_n = n + 1$, which is exactly the result we obtained in Example 1 using induction.

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